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## Tangles, tree-decompositions and grids in matroids <sup>☆</sup>

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### ABSTRACT

A tangle in a matroid is an obstruction to small branch-width. In particular, the maximum order of a tangle is equal to the branch-width. We prove that: (i) there is a tree-decomposition of a matroid that “displays” all of the maximal tangles, and (ii) when  $M$  is representable over a finite field, each tangle of sufficiently large order “dominates” a large grid-minor. This extends results of Robertson and Seymour concerning Graph Minors.

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## 1. Introduction

Robertson and Seymour [6] introduced branch-width for graphs and showed that this parameter is characterized by “tangles”. Robertson and Seymour also stated that their results extend to matroids [6, p. 190]; the details were later given by Dharmatilake [1] (see, also, [3]). Here we use the definitions given in [3]; we defer these definitions until Section 3. For the purpose of this introduction, a tangle of order  $\theta$  in  $M$  can be thought of as a “ $\theta$ -connected component” of  $M$ . We prove the following two results.

### 1.1. Each matroid has a tree-decomposition that “displays” all its maximal tangles.

This will be made precise in Theorem 9.1, which extends a result in Graph Minors X [6, (10.3)].

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**Theorem 1.2.** For each finite field  $\mathbb{F}$  and positive integer  $k$  there exists an integer  $\theta$  such that, if  $M$  is an  $\mathbb{F}$ -representable matroid and  $\mathcal{T}$  is a tangle in  $M$  of order  $\theta$ , then  $\mathcal{T}$  dominates a minor  $N$  that is isomorphic to the cycle matroid of a  $k$  by  $k$  grid.

The proof is given in Section 7. Theorem 1.2 extends a result of Robertson, Seymour, and Thomas [8, (2.3)]. The term “dominates” is used specifically with respect to grid-minors and is defined in Section 7. To prove Theorem 1.2 we will use the main result of [4] which says that an  $\mathbb{F}$ -representable matroid with huge branch-width contains a large grid-minor.

These results are technical, but the motivation is to, hopefully, use them in extending the Graph Minors Structure Theorem [7]. For example, for certain fixed binary matroids  $N$ , we are interested in the class of binary matroids that do not contain an  $N$ -minor. Typically we choose  $N$  to be a highly structured matroid, such as: the cycle matroid of a grid, the cycle matroid of a complete graph, or a projective geometry. In such cases  $N$  has a unique maximal tangle  $\mathcal{T}_N$ . Now, if  $N$  is a minor of some binary matroid  $M$ , then the tangle  $\mathcal{T}_N$  “induces” a tangle  $\mathcal{T}_M$  in  $M$ . Any tangle in  $M$  that contains  $\mathcal{T}_M$  is said to “dominate”  $N$ . Now 1.1 shows that the maximal tangles in  $M$  are composed in a tree-like way. This tree structure essentially localizes each maximal tangle in  $M$  and shows how  $M$  is composed from these local parts. So, to determine the structure of binary matroids with no  $N$ -minor, it suffices to determine the local structure of each maximal tangle in  $M$  that does not dominate an  $N$ -minor. Unfortunately the local structure of tangles that do not dominate  $N$  is complicated. This is partly overcome by considering only tangles whose order is much larger than the order of  $\mathcal{T}_N$ . By Theorem 1.2, each such tangle dominates a huge grid. Supposing that our tangle does not dominate an  $N$ -minor, the hope then is that this huge grid-minor will impose local structure on  $M$ .

## 2. Connectivity and branch-width

We assume that the reader is familiar with matroid theory; we use the notation of Oxley [5].

Let  $\lambda$  be a function that assigns an integer value to each subset of a finite set  $E$ . We call  $\lambda$  *symmetric* if  $\lambda(X) = \lambda(E - X)$  for all  $X \subseteq E$ . We call  $\lambda$  *submodular* if  $\lambda(X) + \lambda(Y) \geq \lambda(X \cap Y) + \lambda(X \cup Y)$  for all  $X, Y \subseteq E$ . If  $\lambda$  is integer-valued, symmetric, and submodular, then we call  $\lambda$  a *connectivity function* on  $E$ . A *connectivity system* is a pair  $K = (E, \lambda)$  where  $\lambda$  is a connectivity function on  $E$ . A partition  $(A, B)$  of  $E(K)$  is called a *separation of order*  $\lambda_K(A)$ .

For a matroid  $M$  and  $X \subseteq E(M)$ , we let  $\lambda_M(X) = r_M(X) + r_M(E(M) - X) - r(M) + 1$ . It is straightforward to prove that  $K_M = (E(M), \lambda_M)$  is a connectivity system. For a graph  $G$  and  $X \subseteq E(G)$ , we let  $\lambda_G(X)$  denote the number of vertices of  $G$  that are incident with both an edge of  $X$  and an edge of  $E(G) - X$ . It is also straightforward to prove that  $K_G = (E(G), \lambda_G)$  is a connectivity system. Moreover, if  $G$  is connected we have for each  $X \subseteq E(G)$  that  $\lambda_{M(G)}(X) \leq \lambda_G(X)$ .

Branch-width plays only a minor role in this paper, but we include a definition for completeness. Let  $K$  be a connectivity system. A tree is *cubic* if its internal vertices all have degree 3. A *branch-decomposition* of  $K$  is a cubic tree  $T$  whose leaves are labeled by elements of  $E(K)$  such that each element in  $E(K)$  labels exactly one leaf of  $T$  and each leaf of  $T$  receives at most one label from  $E(K)$ . If  $T'$  is a subgraph of  $T$  and  $X \subseteq E(K)$  is the set of labels of  $T'$ , then we say that  $T'$  *displays*  $X$ . The *width* of an edge  $e$  of  $T$  is defined to be  $\lambda_K(X)$  where  $X$  is the set displayed by one of the components of  $T - \{e\}$ . The *width* of  $T$  is the maximum among the widths of its edges. The *branch-width* of  $K$  is the minimum among the widths of all branch-decompositions of  $K$ .

The *branch-width* of a matroid  $M$  is the branch-width of its connectivity system  $K_M = (E(M), \lambda_M)$ .

We remark that there are some trivial graphs  $G$ , such as trees, for which  $K_G$  and  $K_{M(G)}$  have different branch-width. It is, however, conjectured that, if  $G$  has a circuit of length at least 2, then  $K_G$  and  $K_{M(G)}$  have the same branch-width. In Section 6 we prove that this is at least true for  $n$  by  $n$  grids.

## 3. Tangles

In this section we review results and definitions from [3].

Let  $K$  be a connectivity system. A *tangle* in  $K$  of order  $\theta$  is a collection  $\mathcal{T}$  of subsets of  $E(K)$  such that:

- (T1) For each  $B \in \mathcal{T}$ ,  $\lambda_K(B) < \theta$ .
- (T2) For each separation  $(A, B)$  of order less than  $\theta$ ,  $\mathcal{T}$  contains either  $A$  or  $B$ .
- (T3) If  $A, B, C \in \mathcal{T}$ , then  $A \cup B \cup C \neq E(K)$ .
- (T4) For each  $e \in E(K)$ ,  $E(K) - \{e\} \notin \mathcal{T}$ .

It is proved in [3, Lemma 3.1] that, to verify that  $\mathcal{T}$  is a tangle, we may replace (T3) by the following weaker conditions:

- (T3a) If  $B \in \mathcal{T}$ ,  $A \subseteq B$ , and  $\lambda_K(A) < \theta$ , then  $A \in \mathcal{T}$ .
- (T3b) If  $(A_1, A_2, A_3)$  is a partition of  $E(K)$ , then  $\mathcal{T}$  does not contain all three of  $A_1, A_2$ , and  $A_3$ .

Note that throughout this text partitions may have empty members; in particular, (T3b) also says that no two members of  $\mathcal{T}$  partition  $E(K)$ .

The following slight variation of [6, (3.5)] was proved in [3, Theorem 3.2].

**Theorem 3.1.** *Let  $K$  be a connectivity system. Then, the maximum order of a tangle in  $K$  is equal to the branch-width of  $K$ .*

A tangle in a matroid  $M$  is a tangle in its connectivity system  $K_M$ . The following fact is used in the proof of 7.3.1.

**Lemma 3.2.** *Let  $\mathcal{T}$  be a tangle of order  $\theta$  at least 3 in a matroid  $M$ . Then each subset of  $E(M)$  with rank less than  $\theta - 1$  is in  $\mathcal{T}$ .*

**Proof.** Let  $X$  be a smallest possible subset in  $E(M)$  that is not in  $\mathcal{T}$ . As  $\theta \geq 3$  it follows from (T2) and (T4) that singletons are in  $\mathcal{T}$ . So  $X$  can be partitioned into two smaller sets. By the choice of  $X$  these two sets are in  $\mathcal{T}$ . Hence by (T3),  $E(M) - X$  is not in  $\mathcal{T}$ . Thus by (T2),  $\lambda_M(X) \geq \theta$ . Note that, for any  $Y \subseteq E(M)$ , the rank of  $Y$  is at least  $\lambda_M(Y) - 1$ . So  $X$  has rank at least  $\theta - 1$ ; as required.  $\square$

Let  $\mathcal{T}$  be a tangle of order  $\theta$  in matroid  $M$ . For  $X \subseteq E(M)$ , if  $X$  is a subset of a set in  $\mathcal{T}$ , then we let

$$\phi_{\mathcal{T}}(X) = \min(\lambda_M(A) - 1 : X \subseteq A \in \mathcal{T}),$$

otherwise we let  $\phi_{\mathcal{T}}(X) = \theta - 1$ . The following result was proved in [3, Lemma 4.3].

**Lemma 3.3.** *Let  $M$  be a matroid and let  $\mathcal{T}$  be a tangle in  $M$  of order  $\theta$ . Then  $\phi_{\mathcal{T}}$  is the rank-function of a matroid of rank  $\theta - 1$ .*

This matroid is referred to as the *tangle matroid* of  $\mathcal{T}$ .

#### 4. New tangles from old

In this section we look at different constructions for tangles. Let  $\mathcal{T}$  be a tangle of order  $\theta$  in a connectivity system  $K$  and let  $\theta' \leq \theta$ . Now let  $\mathcal{T}'$  be the collection of all sets  $A \in \mathcal{T}$  with  $\lambda_K(A) < \theta'$ . It is straightforward to verify that:

**Lemma 4.1.**  *$\mathcal{T}'$  is a tangle in  $K$  of order  $\theta'$ .*

We say that  $\mathcal{T}'$  is the *truncation* of  $\mathcal{T}$  to order  $\theta'$ . Note that if  $\mathcal{T}'$  and  $\mathcal{T}$  are tangles in  $K$ , then  $\mathcal{T}'$  is a truncation of  $\mathcal{T}$  if and only if  $\mathcal{T}' \subseteq \mathcal{T}$ .

Let  $K = (E, \lambda)$  be a connectivity system and let  $X \subseteq E$ . We let  $K \circ X = ((E - X) \cup \{e_X\}, \lambda')$  where, for each  $A \subseteq E - X$ ,  $\lambda'(A) = \lambda(A)$  and  $\lambda'(A \cup \{e_X\}) = \lambda(A \cup X)$ . It is straightforward to verify that:

**Lemma 4.2.** *If  $K$  is a connectivity system and  $X \subseteq E(K)$ , then  $K \circ X$  is a connectivity system.*

We can also obtain a tangle in  $K \circ X$  from a tangle in  $K$ .

**Lemma 4.3.** *Let  $\mathcal{T}$  be a tangle of order  $\theta$  in the connectivity system  $K$  and let  $X \in \mathcal{T}$ . Now let  $\mathcal{T}'$  be the collection of subsets of  $E(K \circ X)$  such that, for  $A \subseteq E(K) - X$ ,  $A \in \mathcal{T}'$  if and only if  $A \in \mathcal{T}$ ; and  $A \cup \{e_X\} \in \mathcal{T}'$  if and only if  $A \cup X \in \mathcal{T}$ . Then  $\mathcal{T}'$  is a tangle of order  $\theta$  in  $K \circ X$ .*

**Proof.** Each of the conditions (T1)–(T4) for  $\mathcal{T}'$  to be a tangle follows directly from the corresponding condition for  $\mathcal{T}$ .  $\square$

A set  $X$  of elements in a connectivity system  $K$  is called *titanic* if each partition  $(A_1, A_2, A_3)$  of  $X$  satisfies  $\lambda_K(A_i) \geq \lambda_K(X)$  for at least one  $i = 1, 2, 3$ .

The following result is a partial converse of Lemma 4.3; it generalizes a result in Graph Minors X [6, (8.3)].

**Lemma 4.4.** *Let  $K$  be a connectivity system, let  $X \subseteq E(K)$  be titanically with  $\lambda_K(X) < \theta$ , and let  $\mathcal{T}'$  be a tangle of order  $\theta$  in  $K \circ X$ . Now let  $\mathcal{T}$  be the collection of all  $A \subseteq E(K)$  such that  $\lambda_K(A) < \theta$  and either  $A - X \in \mathcal{T}'$  or  $(A - X) \cup \{e_X\} \in \mathcal{T}'$ . Then  $\mathcal{T}$  is a tangle of order  $\theta$  in  $K$ .*

**Proof.** Let  $Y = E(K) - X$  and  $L = K \circ X$ . Note that  $\lambda_L(\{e_X\}) = \lambda_L(Y) = \lambda_K(Y) = \lambda_K(X) < \theta$ , so  $\{e_X\} \in \mathcal{T}'$ . By definition,  $\mathcal{T}$  satisfies (T1).

We next prove that  $\mathcal{T}$  satisfies (T2). Consider a separation  $(A, B)$  of order less than  $\theta$  in  $K$ . Since  $X$  is titanically in  $K$ , either  $\lambda_K(X \cap A) \geq \lambda_K(X)$  or  $\lambda_K(X \cap B) \geq \lambda_K(X)$ . By symmetry between  $A$  and  $B$ , we may assume that  $\lambda_K(X \cap A) \geq \lambda_K(X)$ . Then, by submodularity and symmetry of  $\lambda_K$ , we see that  $\lambda_L(Y \cap B) = \lambda_K(Y \cap B) = \lambda_K(A \cup X) \leq \lambda_K(A) + \lambda_K(X) - \lambda_K(A \cap X) \leq \lambda_K(A) < \theta$ . Therefore, as  $\mathcal{T}'$  satisfies (T2), one of  $Y \cap B = B - X$  or  $(Y \cap A) \cup \{e_X\} = (A - X) \cup \{e_X\}$  is in  $\mathcal{T}'$ . Thus,  $\mathcal{T}$  contains  $B$  or  $A$ , as required. So  $\mathcal{T}$  satisfies (T2).

Next consider (T3a). Let  $B \in \mathcal{T}$  and  $A \subseteq B$  with  $\lambda_K(A) < \theta$ . Then, by definition,  $B - X$  is contained in a set in  $\mathcal{T}'$ . Since  $A \subseteq B$ , the union of  $(E(K) - A) - X$ ,  $B - X$  and  $\{e_X\}$  is  $E(L)$ . As  $\{e_X\}$  is in  $\mathcal{T}'$  and as  $\mathcal{T}'$  satisfies (T3), this implies that  $(E(K) - A) - X$  is not contained in a set of  $\mathcal{T}'$ . So,  $E(K) - A \notin \mathcal{T}$ . As  $\lambda_K(A) < \theta$  and as  $\mathcal{T}$  does satisfy (T2) this implies that  $A \in \mathcal{T}$ , as required. So  $\mathcal{T}$  satisfies (T3a).

We next prove by contradiction that  $\mathcal{T}$  satisfies (T3b). Let  $A_1, A_2$ , and  $A_3$  be members of  $\mathcal{T}$  that partition  $E(K)$ . Then each of  $A_1 - X$ ,  $A_2 - X$  and  $A_3 - X$  is contained in a set in  $\mathcal{T}'$ . So, since  $E(L)$  cannot be covered by three sets in  $\mathcal{T}'$ , none of the sets  $(A_1 \cap Y) \cup \{e_X\}$ ,  $(A_2 \cap Y) \cup \{e_X\}$ , or  $(A_3 \cap Y) \cup \{e_X\}$  is in  $\mathcal{T}'$ . Thus  $\mathcal{T}'$  contains each of  $A_1 \cap Y$ ,  $A_2 \cap Y$ , and  $A_3 \cap Y$ . Since  $A_1 \cap Y$  and  $\{e_X\}$  lie in  $\mathcal{T}'$ ,  $\mathcal{T}'$  does not contain  $Y - A_1$ . Now since  $\mathcal{T}'$  contains neither  $Y - A_1$  nor  $(A_1 \cap Y) \cup \{e_X\}$ , we have  $\lambda_K(Y - A_1) = \lambda_L(Y - A_1) \geq \theta > \lambda_K(A_1)$ . So, by submodularity and symmetry of  $\lambda_K$ , we get that  $\lambda_K(X \cap A_1) \leq \lambda_K(X) + \lambda_K(A_1) - \lambda_K(X \cup A_1) = \lambda_K(X) + \lambda_K(A_1) - \lambda_K(Y - A_1) < \lambda_K(X)$ . Similarly  $\lambda_K(X \cap A_2) < \lambda_K(X)$  and  $\lambda_K(X \cap A_3) < \lambda_K(X)$ . However this contradicts the fact that  $X$  is titanically. Thus  $\mathcal{T}$  satisfies (T3b) and, hence,  $\mathcal{T}$  is a tangle of order  $\theta$  in  $K$ .

Finally we prove by contradiction that  $\mathcal{T}$  satisfies (T4). Suppose  $e \in E(K)$  with  $E(K) - \{e\} \in \mathcal{T}$ . Then at least one of  $E(L) - \{e, e_X\} = E(K) - \{e\} - X$  or  $E(L) - \{e\} = (E(K) - \{e\} - X) \cup \{e_X\}$  is in  $\mathcal{T}'$ . As  $\mathcal{T}'$  satisfies (T4), this means  $E(L) - \{e, e_X\} \in \mathcal{T}'$  and  $e \in E(L) - \{e_X\}$ . Now we have, as  $E(K) - \{e\} \in \mathcal{T}$ , that  $\lambda_L(\{e\}) = \lambda_K(\{e\}) = \lambda_K(E(K) - \{e\}) < \theta$ . So, as  $\mathcal{T}'$  satisfies (T4), the singleton  $\{e\}$  is in  $\mathcal{T}'$ . But since also  $\{e_X\}$  and  $E(L) - \{e, e_X\}$  are in  $\mathcal{T}'$ , this contradicts that  $\mathcal{T}'$  satisfies (T3). So  $\mathcal{T}$  does indeed satisfy (T4).  $\square$

## 5. Minors and tangles

Let  $N$  be a minor of  $M$  and let  $\mathcal{T}_N$  be a tangle in  $N$  of order  $\theta$ . Now let  $\mathcal{T}_M$  be the collection of all sets  $A \subseteq E(M)$  where  $\lambda_M(A) < \theta$  and  $A \cap E(N) \in \mathcal{T}_N$ . The following result is an immediate consequence of definitions.

**Lemma 5.1.**  $\mathcal{T}_M$  is a tangle in  $M$  of order  $\theta$ .

We say that  $\mathcal{T}_M$  is the tangle in  $M$  induced by  $\mathcal{T}_N$ .

Let  $f: \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$  be a function and  $m \in \mathbb{Z}_+$ . A matroid  $M$  is called  $(m, f)$ -connected if whenever  $(A, B)$  is a separation of order  $\ell$  where  $\ell < m$  we have either  $|A| \leq f(\ell)$  or  $|B| \leq f(\ell)$ .

Let  $g(n) = (6^{n-1} - 1)/5$ . Note that  $g(1) = 0$  and  $g(n) = 6g(n-1) + 1$  for all  $n > 1$ . The main result in this section is the following.

**Theorem 5.2.** Let  $\mathcal{T}$  be a tangle of order  $\theta$  in a matroid  $M$ . Then there exists a  $(\theta, g)$ -connected minor  $N$  of  $M$  and a tangle  $\mathcal{T}'$  of order  $\theta$  in  $N$  such that  $\mathcal{T}$  is the tangle in  $M$  induced by  $\mathcal{T}'$ .

We will use the following result from [2, Lemma 3.1].

**Lemma 5.3.** Let  $f: \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$  be a nondecreasing function. If  $e$  is an element of an  $(m, f)$ -connected matroid  $M$ , then  $M \setminus e$  or  $M/e$  is  $(m, 2f)$ -connected.

**5.4. Proof of Theorem 5.2.** The proof is by induction on  $|E(M)|$  with  $\theta$  fixed; the root of this induction lies in the  $(\theta, g)$ -connected matroids. Let  $\mathcal{T}$  be a tangle of order  $\theta$  in a matroid  $M$  and assume  $M$  is not  $(\theta, g)$ -connected. Choose  $m \in \{1, \dots, \theta - 1\}$  as small as possible such that  $M$  is not  $(m + 1, g)$ -connected. Then there exists a separation  $(A, B)$  of order  $m$  with  $|A|, |B| > g(m)$ . By symmetry we may assume that  $A \in \mathcal{T}$ . Now let  $e \in A$ . By Lemma 5.3 and duality, we may assume that  $M/e$  is  $(m, 2g)$ -connected.

**5.4.1.**  $A - \{e\}$  is titanic in  $M/e$ .

**Subproof.** When  $m = 1$  this is vacuously true. Suppose that  $m > 1$  and consider any partition  $(A_1, A_2, A_3)$  of  $A - \{e\}$ . Since  $|A| > g(m) = 6g(m-1) + 1$ , we have  $|A_i| > 2g(m-1)$  for some  $i \in \{1, 2, 3\}$ . Then, since  $M/e$  is  $(m, 2g)$ -connected,  $\lambda_{M/e}(A_i) \geq m \geq \lambda_{M/e}(A - \{e\})$ . Hence  $A - \{e\}$  is indeed titanic in  $M/e$ .  $\square$

**5.4.2.** For each  $X \subseteq B$ ,  $\lambda_M(X) = \lambda_{M/e}(X)$ .

**Subproof.** Since  $M/e$  is  $(m, 2g)$ -connected,  $\lambda_M(B) = \lambda_{M/e}(B)$ . Hence  $e \notin \text{cl}_M(B)$ . Therefore, for each  $X \subseteq B$ ,  $e \notin \text{cl}_M(X)$ . So  $\lambda_M(X) = \lambda_{M/e}(X)$ ; as required.  $\square$

**5.4.3.** For each  $X \subseteq E(M)$  with  $\lambda_M(X) < \theta$  we have that  $X \in \mathcal{T}$  if and only if  $X - A \in \mathcal{T}$  or  $X \cup A \in \mathcal{T}$ .

**Subproof.** Let  $X \subseteq E(M)$  with  $\lambda_M(X) < \theta$ . First assume that  $X - A \in \mathcal{T}$  or  $X \cup A \in \mathcal{T}$ . Then, as  $A \in \mathcal{T}$ , it follows from (T3) that  $E(M) - X \notin \mathcal{T}$ . Hence  $X \in \mathcal{T}$ .

For the reverse implication assume now that  $X \in \mathcal{T}$ . By 5.4.2,  $\lambda_M(A) = \lambda_M(B) = \lambda_{M/e}(B - \{e\}) = \lambda_{M/e}(A - \{e\})$ . So as  $A$  is titanic in  $M/e$  either  $\lambda_M(A - X) \geq \lambda_{M/e}(A - X) \geq \lambda_M(A)$  or  $\lambda_M(A \cup X) \geq \lambda_{M/e}(A \cup X) \geq \lambda_M(A)$ . If  $\lambda_M(A - X) \geq \lambda_M(A)$ , then by symmetry and submodularity of  $\lambda_M$  we have that  $\lambda_M(X - A) = \lambda_M(X \cap B) \leq \lambda_M(X) + \lambda_M(B) - \lambda_M(X \cup B) = \lambda_M(X) + \lambda_M(A) - \lambda_M(A - X) \leq \lambda_M(X) < \theta$ . Hence, if  $\lambda_M(A - X) \geq \lambda_M(A)$  then it follows from (T3a) that  $X - A \in \mathcal{T}$ . If  $\lambda_M(A \cap X) \geq \lambda_M(A)$ , then, again by submodularity,  $\lambda_M(A \cup X) \leq \lambda_M(X) + \lambda_M(A) - \lambda_M(A \cap X) \leq \lambda_M(X) < \theta$ . So by (T2) either  $A \cup X \in \mathcal{T}$  or  $B - X \in \mathcal{T}$ . However, as  $A \in \mathcal{T}$  and  $X \in \mathcal{T}$  it follows from (T3) that  $B - X \notin \mathcal{T}$ . So  $A \cup X \in \mathcal{T}$ . We conclude that if  $X \in \mathcal{T}$  then  $X - A \in \mathcal{T}$  or  $X \cup A \in \mathcal{T}$ .  $\square$

Let  $\mathcal{T}_1$  be the tangle in  $K_M \circ A$  of order  $\theta$  obtained from  $\mathcal{T}$  via Lemma 4.3. By 5.4.2, there is a natural isomorphism between  $K_M \circ A$  and  $K_{M/e} \circ (A - \{e\})$ ; let  $\mathcal{T}_2$  be the tangle in  $K_{M/e} \circ (A - \{e\})$  of order  $\theta$  that is obtained from  $\mathcal{T}_1$  via this isomorphism. In both  $K_M \circ A$  and  $K_{M/e} \circ (A - \{e\})$  denote the element that is not in  $B$  by  $e'$ .

Let  $\mathcal{T}_3$  be the tangle in  $M/e$  of order  $\theta$  that is obtained from  $\mathcal{T}_2$  via Lemma 4.4. Finally let  $\mathcal{T}_4$  be the tangle in  $M$  that is induced by  $\mathcal{T}_3$ .

**5.4.4.**  $\mathcal{T} = \mathcal{T}_4$ .

**Subproof.** Let  $(X, Y)$  be a separation of  $M$  of order less than  $\theta$  with  $e \in Y$ . Then each of the following sequence of equivalences follows directly from definitions:

$$\begin{aligned} X \in \mathcal{T}_4 &\iff X \in \mathcal{T}_3 \\ &\iff X - (A - \{e\}) \in \mathcal{T}_2 \text{ or } (X - (A - \{e\})) \cup \{e'\} \in \mathcal{T}_2 \\ &\iff X - A \in \mathcal{T}_1 \text{ or } (X - A) \cup \{e'\} \in \mathcal{T}_1 \\ &\iff X - A \in \mathcal{T} \text{ or } X \cup A \in \mathcal{T}. \end{aligned}$$

So by 5.4.3,  $X \in \mathcal{T}_4$  if and only if  $X \in \mathcal{T}$ ; as required.  $\square$

The result now follows easily by applying induction to the tangle  $\mathcal{T}_3$  in  $M/e$ .  $\square$

**6. A tangle in a grid**

An  $n$  by  $n$  grid is a graph  $G_n$  with vertex set  $V = \{(i, j) : i, j \in \{1, \dots, n\}\}$  where vertices  $(i, j)$  and  $(i', j')$  are adjacent if and only if either  $i = i'$  and  $|j - j'| = 1$ , or  $j = j'$  and  $|i - i'| = 1$ .

The goal of this section is to prove the existence of a natural tangle of order  $n$  in  $M(G_n)$ . For  $i \in \{1, \dots, n\}$  let  $P_i$  denote the path in  $G_n$  on vertices  $(i, 1), \dots, (i, n)$  and let  $Q_i$  denote the path in  $G_n$  on vertices  $(1, i), \dots, (n, i)$ . Now we let  $\mathcal{T}_n$  denote the collection of all subsets  $A \subseteq E(G_n)$  such that  $\lambda_{M(G_n)}(A) < n$  and  $A$  does not contain any  $E(P_i)$  for  $i \in \{1, \dots, n\}$ . We will prove, for  $n \geq 3$ :

**Lemma 6.1.**  $\mathcal{T}_n$  is a tangle in  $M(G_n)$  of order  $n$ .

A similar result was proved by Kleitman and Saks; see [6, (7.3)]. They considered tangles in  $K_{G_n}$ , whereas we consider tangles in  $K_{M(G_n)}$ . Our proof follows that of Kleitman and Saks; we need some preliminary results on connectivity.

Let  $X$  and  $Y$  be disjoint subsets of  $E(M)$ , we define  $\kappa_M(X, Y) = \min(\lambda_M(A) : X \subseteq A \subseteq E(M) - Y)$ . The following result, due to Tutte [9], is an extension of Menger’s Theorem.

**Theorem 6.2 (Tutte’s Linking Theorem).** *If  $S$  and  $T$  are disjoint sets of elements in a matroid  $M$ , then there exists a minor  $N$  of  $M$  such that  $E(N) = S \cup T$  and  $\lambda_N(S) = \kappa_M(S, T)$ .*

The following result was proved in [4].

**Lemma 6.3.** *Let  $S$  and  $T$  be disjoint sets of elements of a matroid  $M$ . Then there exist sets  $S_1 \subseteq S$  and  $T_1 \subseteq T$  such that  $|S_1| + 1 = |T_1| + 1 = \kappa_M(S_1, T_1) = \kappa_M(S, T)$ .*

In order to prove Lemma 6.1, we first need to establish that certain sets of edges in a grid are “highly connected”.

**Lemma 6.4.** *Let  $i \in \{1, \dots, n\}$  and, for each  $j \in \{1, \dots, n\} - \{i\}$ , let  $e_j$  and  $f_j$  be disjoint edges of  $P_j$ . Now let  $X = \{e_j : j \in \{1, \dots, n\} - \{i\}\}$  and let  $Y = \{f_j : j \in \{1, \dots, n\} - \{i\}\}$ . Then  $\kappa_{M(G_n)}(X, Y) = n$ .*

**Proof.** Let  $D = E(Q_2) \cup \dots \cup E(Q_{n-1})$  and let  $C = E(Q_1) \cup E(Q_n) \cup ((E(P_1) \cup \dots \cup E(P_n)) - (X \cup Y))$ . Now let  $H = G_n \setminus D/C$ . Note that  $H[X]$  and  $H[Y]$  are disjoint spanning trees of  $H$ . Therefore  $n = \lambda_{M(H)}(X) = \kappa_{M(H)}(X, Y) \leq \kappa_{M(G_n)}(X, Y) \leq |X| + 1 = n$ . Thus  $\kappa_{M(G_n)}(X, Y) = n$ , as required.  $\square$

The proofs of the following two results are similar to that of Lemma 6.4; we leave these to the reader.

**Lemma 6.5.** *Let  $i, j \in \{1, \dots, n\}$ . Then  $\kappa_{M(G_n)}(P_i, Q_j) = n$ . Also, if  $i \neq j$ , then  $\kappa_{M(G_n)}(P_i, P_j) = n$  and  $\kappa_{M(G_n)}(Q_i, Q_j) = n$ .*

**Lemma 6.6.** *Let  $X \subseteq E(P_1) \cup E(P_n)$  with  $|X| \geq n - 1$  and let  $j \in \{1, \dots, n\}$ . Then  $\kappa_{M(G_n)}(X, Q_j) = n$ .*

We call a set  $A \subseteq E(G_n)$  *small* if  $\lambda_{M(G_n)}(A) < n$  and  $A$  does not contain any of  $E(P_1), \dots, E(P_n)$  or  $E(Q_1), \dots, E(Q_n)$ .

**Lemma 6.7.** *Let  $(A, B)$  be a separation of  $M(G_n)$  of order less than  $n$ . Then one of  $A$  and  $B$  is small. Moreover, if  $B$  is small, then  $A$  contains one of  $E(P_1), \dots, E(P_n)$  and one of  $E(Q_1), \dots, E(Q_n)$ .*

**Proof.** By Lemma 6.4, either  $A$  or  $B$  must contain one of  $E(P_1), \dots, E(P_n)$ . Then, by symmetry, either  $A$  or  $B$  must contain one of  $E(Q_1), \dots, E(Q_n)$ . However, by Lemma 6.5,  $A$  and  $B$  cannot both contain one of  $E(P_1), \dots, E(P_n), E(Q_1), \dots, E(Q_n)$ .  $\square$

Note that  $\mathcal{T}_n$  trivially satisfies conditions (T1), (T3a), and (T4). By Lemma 6.7,  $\mathcal{T}_n$  also satisfies (T2). Thus in order to complete the proof of Lemma 6.1, we need only verify (T3b); this is achieved by the following result.

**Lemma 6.8.** *For  $n \geq 3$ ,  $E(G_n)$  cannot be partitioned into three small sets.*

**Proof.** The proof is by induction on  $n$ . The case  $n = 3$  is trivial; suppose then that  $n \geq 4$  and that the result holds for  $G_{n-1}$ . Now assume  $(A_1, A_2, A_3)$  is a partition of  $E(G_n)$  into three small sets.

By symmetry we may assume that  $Q_n$  meets  $A_1$  and  $A_2$ . (That is,  $A_1 \cap E(Q_n)$  and  $A_2 \cap E(Q_n)$  are nonempty.) By Lemma 6.7, there is a path  $Q_j$  disjoint from  $A_1$ . Note that  $\kappa_{M(G_n)}(A_1 \cap (E(P_1) \cup E(P_n)), Q_j) \leq \lambda_{M(G_n)}(A_1) < n$ . Then, by Lemma 6.6,  $|A_1 \cap (E(P_1) \cup E(P_n))| < n - 1$ . Similarly  $|A_2 \cap (E(P_1) \cup E(P_n))| < n - 1$ . Therefore either  $P_1$  or  $P_n$  meets  $A_3$ ; by symmetry, we may assume that  $P_n$  meets  $A_3$ . Therefore  $E(P_n) \cup E(Q_n)$  meets each of  $A_1, A_2$ , and  $A_3$ .

Note that  $G_{n-1} = G_n - (V(P_n) \cup V(Q_n))$ . For each  $i \in \{1, 2, 3\}$ , let  $A'_i = E(G_{n-1}) \cap A_i$ .

**6.8.1.** *There exists  $k \in \{1, 2, 3\}$  such that  $\lambda_{M(G_{n-1})}(A'_k) \geq n - 1$ .*

**Subproof.** By the induction hypothesis, there exists  $k \in \{1, 2, 3\}$  such that  $A'_k$  is not small in  $G_{n-1}$ . Suppose that  $\lambda_{M(G_{n-1})}(A'_k) < n - 1$ . Then  $A'_k$  contains one of  $E(P_1) \cap E(G_{n-1}), \dots, E(P_{n-1}) \cap E(G_{n-1})$  or one of  $E(Q_1) \cap E(G_{n-1}), \dots, E(Q_{n-1}) \cap E(G_{n-1})$ . By Lemma 6.7,  $A_k$  avoids some path  $P_i$  and some path  $Q_j$ . Since  $E(P_n) \cup E(Q_n)$  meets each of  $A_1, A_2$ , and  $A_3$ , either  $i \neq n$  or  $j \neq n$ . Thus  $A'_k$  avoids one of  $E(P_1) \cap E(G_{n-1}), \dots, E(P_{n-1}) \cap E(G_{n-1})$  or one of  $E(Q_1) \cap E(G_{n-1}), \dots, E(Q_{n-1}) \cap E(G_{n-1})$ . So, applying Lemma 6.7 to  $G_{n-1}$ , we contradict the assumption that  $\lambda_{M(G_{n-1})}(A'_k) < n - 1$ .  $\square$

By Lemma 6.3, there exist  $S \subseteq A'_k$  and  $T \subseteq E(G_{n-1}) - A'_k$  such that  $|S| + 1 = |T| + 1 = \kappa_{M(G_{n-1})}(S, T) \geq n - 1$ . Now, by Tutte's Linking Theorem, there exists a minor  $H$  of  $G_{n-1}$  such that  $E(H) = S \cup T$  and  $\lambda_{M(H)}(S) \geq n$ . Suppose that  $H = G_{n-1} \setminus D/C$ ; we may choose  $D$  and  $C$  such that  $D$  does not contain a cut of  $G_n$ . Thus  $H$  is connected and  $S$  and  $T$  are disjoint spanning trees of  $H$ ; thus  $|V(H)| \geq n - 1$ . Now let  $H' = G_n \setminus D/H$ . Vertices  $(1, n)$  and  $(n, 1)$  both have a neighbour in  $V(H)$  in  $H'$ . Note that there exist  $e \in (E(P_n) \cup E(Q_n)) \cap A_k$  and  $f \in (E(P_n) \cup E(Q_n)) - A_k$ . Now there exists a minor  $H''$  of  $H'$  such that  $S \cup \{e\}$  and  $T \cup \{f\}$  are disjoint spanning trees of  $H''$ . Thus  $\lambda_{M(H'')}(S \cup \{f\}) \geq n$ . However, this contradicts the fact that  $\lambda_M(A_k) < n$ .  $\square$

**7. A grid in a tangle**

Let  $M$  be a matroid and let  $N$  be a minor of  $M$  that is isomorphic to the cycle matroid of the  $n$  by  $n$  grid. Now let  $\mathcal{T}_N$  be the tangle in  $N$  of order  $n$  given by Lemma 6.1 and let  $\mathcal{T}_M$  be the tangle in  $M$  of order  $n$  that is induced by  $\mathcal{T}_N$ . (We recall that the term “induced” was defined at the start of Section 5 and the term “truncation” was defined at the start of Section 4.) A tangle  $\mathcal{T}$  in  $M$  is said to *dominate*  $N$  if  $\mathcal{T}_M$  is a truncation of  $\mathcal{T}$ . In this section we prove Theorem 1.2. We need the following lemma. (We use the “tangle matroid” which is defined at the end of Section 3.)

**Lemma 7.1.** *Let  $\mathcal{T}$  be a tangle in a matroid  $M$  and let  $M_{\mathcal{T}}$  be the tangle matroid of  $\mathcal{T}$ . Now let  $G_n$  be the  $n$  by  $n$  grid and suppose that  $N = M(G_n)$  is a minor of  $M$ . Then  $\mathcal{T}$  dominates  $N$  if and only if each of the sets  $E(P_1), \dots, E(P_n)$  is independent in  $M_{\mathcal{T}}$ .*

**Proof.** Note that, if  $\mathcal{T}'$  is the truncation of  $\mathcal{T}$  to order  $n$ , then  $M_{\mathcal{T}'}$  is the truncation of  $M_{\mathcal{T}}$  to rank  $n - 1$ . Thus, by possibly truncating, we may assume that  $\mathcal{T}$  has order  $n$ . Now let  $\mathcal{T}_N$  be the tangle in  $N$  of order  $n$  given by Lemma 6.1 and let  $\mathcal{T}_M$  be the tangle in  $M$  of order  $n$  that is induced by  $\mathcal{T}_N$ . Thus  $\mathcal{T}$  dominates  $N$  if and only if  $\mathcal{T} = \mathcal{T}_M$ . Now  $\mathcal{T} \neq \mathcal{T}_M$  if and only if there exists a set  $A \in \mathcal{T}$  that contains one of  $E(P_1), \dots, E(P_n)$ . On the other hand,  $E(P_i)$  is independent in  $M_{\mathcal{T}}$  if and only if there does not exist  $A \in \mathcal{T}$  such that  $E(P_i) \subseteq A$ .  $\square$

We also need the following result from [4].

**Theorem 7.2.** *There exists an integer-valued function  $f(k, q)$  such that for any positive integer  $k$  and prime-power  $q$ , if  $M$  is a  $GF(q)$ -representable matroid with branch-width at least  $f(k, q)$ , then  $M$  contains a minor isomorphic to  $M(G_k)$ .*

Note that, if  $M$  has a tangle of high order, then  $M$  has large branch-width and, hence by Theorem 7.2,  $M$  has a big grid as a minor. Unfortunately, this grid-minor need not be dominated by the tangle.

**7.3. Proof of Theorem 1.2.** Let  $g(t) = (6^t - 1)/5$  for any integer  $t \geq 0$ . Let  $n = g(k - 1) + 2$ , let  $q$  be the order of  $\mathbb{F}$ , and let  $\theta = f(n, q)$ . Now let  $M$  be an  $\mathbb{F}$ -representable matroid and let  $\mathcal{T}$  be a tangle in  $M$  of order  $\theta$ . By Theorem 5.2, there exists a  $(\theta, g)$ -connected minor  $M_1$  of  $M$  and a tangle  $\mathcal{T}_1$  in  $M_1$  of order  $\theta$  such that  $\mathcal{T}$  is the tangle in  $M$  that is induced by  $\mathcal{T}_1$ . By Theorems 3.1 and 7.2, there exists a minor  $N$  of  $M_1$  that is isomorphic to  $M(G_n)$ . By possibly relabeling, we may assume that  $N = M(G_n)$ . Now let  $P_1, \dots, P_n$  be the vertical paths in  $G_n$ , let  $M_{\mathcal{T}_1}$  be the tangle matroid of  $\mathcal{T}_1$ , and let  $\phi_1$  be the rank-function of  $M_{\mathcal{T}_1}$ .

**7.3.1.**  $\phi_1(E(P_i)) \geq k - 1$  for each  $i \in \{1, \dots, n\}$ .

**Subproof.** Suppose to the contrary that  $\phi_1(E(P_i)) < k - 1$  for some  $i$ . Thus there exists  $A \in \mathcal{T}_1$  such that  $E(P_i) \subseteq A$  and  $\lambda_{M_1}(A) \leq k - 1$ . By definition  $|A| \geq |E(P_i)| = n - 1 > g(k - 1)$ . Therefore, since  $M_1$  is  $(\theta, g)$ -connected,  $|E(M_1) - A| \leq g(k - 1) = n - 2 \leq f(n, q) - 2 < \theta - 1$ . Moreover, as  $k \geq 1$ , we have that  $\theta \geq 3$ . Hence by Lemma 3.2,  $E(M_1) - A \in \mathcal{T}_1$ ; contradicting (T3).  $\square$

For each  $i \in \{1, \dots, k\}$ , let  $A_i$  be an  $M_{\mathcal{T}_1}$ -independent subset of  $E(P_{1+(i-1)k})$  with  $|A_i| = k - 1$ ; as  $k^2 - k + 1 \leq n$  these sets  $A_i$  exist. Now there exists a minor  $H$  of  $G_n$  such that  $H$  is isomorphic to  $G_k$  and such that  $A_1, \dots, A_k$  are the edge-sets of the vertical paths in  $H$ . By Lemma 7.1,  $\mathcal{T}_1$  dominates  $H$ . Then, since  $\mathcal{T}$  is induced by  $\mathcal{T}_1$ ,  $\mathcal{T}$  also dominates  $H$ .  $\square$



### 8. Tree-decompositions and laminar families

We begin by reviewing some elementary results on laminar families and tree-decompositions. Let  $E$  be a set. A partition of  $E$  into two sets is called a *separation* of  $E$ . Two separations  $(A_1, A_2)$  and  $(B_1, B_2)$  of a set  $E$  are said to *cross* if  $A_i \cap B_j \neq \emptyset$  for each  $i$  and  $j$  in  $\{1, 2\}$ . A collection  $\mathcal{S}$  of separations of  $E$  is *laminar* if no two separations in  $\mathcal{S}$  cross.

A *tree-decomposition* of  $E$  consists of a pair  $(T, \mathcal{P})$  where  $T$  is a tree and  $\mathcal{P} = (P_v : v \in V(T))$  is a partition of  $E$  (where one or more of the  $P_v$  may be empty). For any  $X \subseteq V(T)$ , we let  $\mathcal{P}[X]$  denote the set  $\bigcup_{v \in X} P_v$ . Now, for any  $e \in E(T)$ , the *separation of  $E$  displayed by  $e$*  is  $(\mathcal{P}[V(T_1)], \mathcal{P}[V(T_2)])$  where  $T_1$  and  $T_2$  are the two components of  $T - e$ . The following result is both easy and well known.

**Lemma 8.1.** *If  $(T, \mathcal{P})$  is a tree-decomposition of  $E$ , then the set of all separations displayed by  $(T, \mathcal{P})$  is laminar.*

Let  $(T, \mathcal{P})$  be a tree-decomposition of  $E$  and let  $\mathcal{S}$  be a set of separations of  $E$ . We say that  $(T, \mathcal{P})$  *represents  $\mathcal{S}$*  if  $\mathcal{S}$  is the set of separations displayed by  $(T, \mathcal{P})$ . The following converse to Lemma 8.1 is also well known.

**Lemma 8.2.** *If  $\mathcal{S}$  is a laminar set of separations of  $E$ , then there is a tree-decomposition of  $E$  that represents  $\mathcal{S}$ .*

Let  $K$  be a connectivity system. A set  $X \subseteq E(K)$  is *robust* if for each proper partition  $(X_1, X_2)$  of  $X$  either  $\lambda_K(X_1) > \lambda_K(X)$  or  $\lambda_K(X_2) > \lambda_K(X)$ . (A partition is *proper* if all its members are nonempty.) A separation  $(X, Y)$  of  $K$  is *robust* if  $X$  and  $Y$  are both robust.

**Lemma 8.3.** *Let  $K$  be a connectivity system and let  $\mathcal{S}$  be the set of all robust separations of  $K$ . Then  $\mathcal{S}$  is laminar.*

**Proof.** Suppose that  $(A_1, A_2), (B_1, B_2) \in \mathcal{S}$  cross. By symmetry, we may assume that  $\lambda_K(A_1) \leq \lambda_K(B_1)$ . As  $\lambda_K$  is symmetric, we may assume that  $\lambda_K(A_2 \cap B_2) \geq \lambda_K(A_1 \cap B_2)$ ; otherwise swap  $A_1$  and  $A_2$ . Then, since  $B_2$  is robust,  $\lambda_K(A_2 \cap B_2) > \lambda_K(B_2)$ . So symmetry and submodularity of  $\lambda_K$  yield  $\lambda_K(A_1 \cap B_1) \leq \lambda_K(A_1) + \lambda_K(B_1) - \lambda_K(A_1 \cup B_1) = \lambda_K(A_1) + \lambda_K(B_2) - \lambda_K(A_2 \cap B_2) < \lambda_K(A_1)$ . So, since  $A_1$  is robust,  $\lambda_K(A_1 \cap B_2) > \lambda_K(A_1)$ . Also, as  $\lambda_K(B_1) \geq \lambda_K(A_1) \geq \lambda_K(A_1 \cap B_1)$  and as  $B_1$  is robust,  $\lambda_K(A_2 \cap B_1) > \lambda_K(B_1)$ . Combining the last two strict inequalities we get  $\lambda_K(A_1 \cap B_2) + \lambda_K(A_2 \cap B_1) > \lambda_K(A_1) + \lambda_K(B_1) = \lambda_K(A_1) + \lambda_K(B_2)$ . As  $\lambda_K(A_2 \cap B_1) = \lambda_K(A_1 \cup B_2)$ , this contradicts submodularity.  $\square$

### 9. Tree-representations of maximal tangles

The main result of this section is Theorem 9.1; when applied to the maximal tangles  $\mathcal{T}_1, \dots, \mathcal{T}_n$  of the matroid, those that are not truncations of others, it is the result alluded to in the introduction by 1.1.

If  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are two tangles in a connectivity system  $K$ , neither of which is a truncation of the other, then there exists a *distinguishing separation*  $(X_1, X_2)$  with  $X_1 \in \mathcal{T}_1$  and  $X_2 \in \mathcal{T}_2$ .

**Theorem 9.1.** *Let  $K$  be a connectivity system and let  $\mathcal{T}_1, \dots, \mathcal{T}_n$  be tangles in  $K$ , none of which is a truncation of another. Then there exists a tree-decomposition  $(T, \mathcal{P})$  of  $E(K)$  such that  $V(T) = \{1, \dots, n\}$  and such that the following hold:*

- (i) *For each  $i \in V(T)$  and  $e \in E(T)$  if  $T'$  is the component of  $T - e$  containing  $i$  then  $\mathcal{P}[V(T')]$  is not in  $\mathcal{T}_i$ .*
- (ii) *For each pair of distinct vertices  $i$  and  $j$  of  $T$ , there exists a minimum-order distinguishing separation for  $\mathcal{T}_i$  and  $\mathcal{T}_j$  that is displayed by  $T$ .*

Let  $K$  and  $K'$  be connectivity systems with  $E(K) = E(K')$ . We call  $K'$  a *tie-breaker* for  $K$  if for each  $X, Y \subseteq E(K)$ :

- (i)  $\lambda_{K'}(X) \neq \lambda_{K'}(Y)$  unless  $X = Y$  or  $X = E(K) - Y$ ,
- (ii)  $\lambda_{K'}(X) < \lambda_{K'}(Y)$  if  $\lambda_K(X) < \lambda_K(Y)$ .

**Lemma 9.2.** *Each connectivity system has a tie-breaker.*

**Proof.** Let  $K$  be a connectivity system. We may assume that  $E(K) = \{1, \dots, n\}$ . Now, for  $X \subseteq \{1, \dots, n - 1\}$ , let  $\lambda_L(X) = \sum_{i \in X} 2^i$  and let  $\lambda_L(E(K) - X) = \lambda_L(X)$ . We leave it to the reader to verify that  $L = (E(K), \lambda_L)$  is indeed a connectivity system. Now, for each  $X \subseteq E(K)$ , we let  $\lambda_{K'}(X) = 2^n \lambda_K(X) + \lambda_L(X)$ . It is easy to check that  $K' = (E(K), \lambda_{K'})$  has the desired properties.  $\square$

It is evident that a tangle in a connectivity system  $K$  is a tangle in any tie-breaker for  $K$ .

**Lemma 9.3.** *Let  $\mathcal{T}_1$  and  $\mathcal{T}_2$  be tangles in a connectivity system  $K$  that are incomparable by truncation, let  $K'$  be a tie-breaker for  $K$ , and let  $(X_1, X_2)$  be a distinguishing separation for  $\mathcal{T}_1$  and  $\mathcal{T}_2$  with minimum order in  $K'$ . Then  $(X_1, X_2)$  is a robust separation of  $K'$ .*

**Proof.** Suppose otherwise. Then, by symmetry, we may assume that there exists a proper partition  $(A, B)$  of  $X_1$  such that  $\lambda_{K'}(A) \leq \lambda_{K'}(X_1)$  and  $\lambda_{K'}(B) \leq \lambda_{K'}(X_1)$ . Since  $K'$  is a tie-breaker,  $\lambda_{K'}(A) < \lambda_{K'}(X_1)$  and  $\lambda_{K'}(B) < \lambda_{K'}(X_1)$ . Condition (T3a) for  $\mathcal{T}_1$  implies that  $A, B \in \mathcal{T}_1$ . Then, by our choice of the distinguishing separation  $(X_1, X_2)$ ,  $\mathcal{T}_2$  contains neither  $E(K) - A$  nor  $E(K) - B$ . Then, by (T2),  $A, B \in \mathcal{T}_2$ . But then  $\mathcal{T}_2$  contains each of  $A, B$ , and  $X_2$ ; contrary to (T3).  $\square$

**Proof of Theorem 9.1.** Let  $K'$  be a tie-breaker for  $K$ . As  $\mathcal{T}_1, \dots, \mathcal{T}_n$  are tangles in  $K'$ , we may assume that  $K = K'$ . For each  $i, j \in \{1, \dots, n\}$  with  $i \neq j$  let  $(X_{ij}, Y_{ij})$  be the minimum-order separation of  $K$  distinguishing  $\mathcal{T}_i$  and  $\mathcal{T}_j$  (where we assume that  $X_{ij} \in \mathcal{T}_i$ ). By Lemma 9.3,  $(X_{ij}, Y_{ij})$  is a robust separation of  $K$ . Now let  $\mathcal{S}$  be the collection of all of these distinguishing separations. By Lemma 8.3,  $\mathcal{S}$  is laminar. Then, by Lemma 8.2, there is a tree-decomposition  $(T, \mathcal{P})$  of  $E(K)$  that represents  $\mathcal{S}$ . We may assume that if  $v$  is a vertex of  $T$  with degree 1 or 2, then  $P_v \neq \emptyset$  (since, otherwise, we could find a smaller tree-decomposition representing  $\mathcal{S}$ ). This means that the edges of  $T$  display proper and distinct separations. It remains to show that there is a bijection between  $\mathcal{T}_1, \dots, \mathcal{T}_n$  and  $V(T)$  satisfying the conclusion of Theorem 9.1.

For  $i = \{1, \dots, n\}$ , consider the collection  $\mathcal{X}_i$  of nonempty subsets  $X$  of  $V(T)$  such that  $E(K) - \mathcal{P}[X] \in \mathcal{T}_i$  and such that  $(\mathcal{P}[X], E(K) - \mathcal{P}[X])$  is displayed by  $T$ . Each member of  $\mathcal{X}_i$  induces a subtree of  $T$  and by (T3) each two members of  $\mathcal{X}_i$  intersect. As any collection of pairwise intersecting subtrees of a tree has a common vertex, the members of  $\mathcal{X}_i$  have a nonempty intersection. Call that intersection  $V_i$ .

Note that by construction of  $V_i$  each edge of  $T$  that leaves  $V_i$  displays a separation  $(A, B)$  with  $\mathcal{P}[V_i] \subseteq A$  and  $B \in \mathcal{T}_i$ . From this, (T2), (T3) and the fact that each separation in  $\mathcal{S}$  is displayed by  $T$  it is straightforward to see that to prove Theorem 9.1 it suffices to show that  $(V_1, \dots, V_n)$  is a partition of  $V(T)$  into singletons.

The sets  $V_1, \dots, V_n$  are pairwise disjoint as for each  $i \neq j$  the set  $\mathcal{P}[V_i]$  lies in  $Y_{ij}$  and the set  $\mathcal{P}[V_j]$  lies in  $Y_{ji} = X_{ij}$ .

It remains to prove that if  $w$  in  $V(T)$  then  $\{w\} = V_i$  for some  $i$ . Among the edges incident with  $w$  take the one that displays the separation,  $(X_{ij}, Y_{ij})$  say, of largest order. So that order is at most the order of  $\mathcal{T}_i$  and of  $\mathcal{T}_j$ . We may assume that  $\mathcal{P}_w \subseteq Y_{ij}$ . As no two edges of  $T$  display the same separation, all other edges incident with  $w$  display a separation of order less than those of  $\mathcal{T}_i$  and  $\mathcal{T}_j$ . By the definition of  $(X_{ij}, Y_{ij})$  these separations do not distinguish  $\mathcal{T}_i$  from  $\mathcal{T}_j$ . Combining that with (T3) for  $\mathcal{T}_j$ , we see that for each of these separations  $\mathcal{P}_w$  is not part of the side that is in  $\mathcal{T}_i$ . Hence  $V_i \subseteq \{w\}$ . As  $V_i$  is not empty,  $\{w\} = V_i$  as claimed.  $\square$

We conclude with a simple corollary to Theorem 9.1.

**Corollary 9.4.** *An  $m$ -element connectivity system has at most  $\frac{m-2}{2}$  maximal tangles.*

**Proof.** Let  $K$  be an  $m$ -element connectivity system and let  $\mathcal{T}_1, \dots, \mathcal{T}_n$  be the maximal tangles in  $K$ . Now let  $(T, \mathcal{P})$  be the tree-decomposition of  $E(M)$  given by Theorem 9.1. Let  $v$  be a vertex of  $T$  of degree  $d_v$ . By (T3) and (T4),  $d_v + |P_v| \geq 4$ . Now  $4n \leq \sum_{i=1}^n (d_i + |P_i|) = 2|E(T)| + |E(M)| = 2(n-1) + m$ . So  $n \leq \frac{m-2}{2}$  as claimed.  $\square$

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